Dynamics of random sparse matrix network topology

Takako Hoshiyama
Department of Advanced Interdisciplinary Studies
The University of Tokyo
Japan
thoshiyama@g.ecc.u-tokyo.ac.jp

Hironori Shimoyama
Professional of Computational Science and Education
Japan
shim9213@gmail.com

***Abstract*— In this paper, we envisage several types of validation models for a more realistic modelling analysis of disaster risk management. Therefore we extend our previous work on the analysis of the characteristics of disaster prevention behavior in the centralized invocation of n-dimensional networks which is based on the free-end spring oscillating systems as random Laplacian matrices. Through this work, we have identified physical properties in several model types. As an advantage of this study over other existing studies, the physical properties of the sparse matrix network associated with specific topology are explained, although the sparsity of the matrices is the same regardless of the graph structure, the type of topology produces characteristic differences in the eigenvalue distributions and eigenvectors. We also notice the distinctive differences in the eigenvectors as the stationary states corresponding to the largest eigenvalues. The analytical methods and potential applications of random Laplacian sparse matrices will also be discussed.**

Keywords—echo chamber, random Laplacian, network dynamics, sparse network

I. INTRODUCTION

Sparse matrices, the focus of this paper, are typically found in network representations such as adjacency and Laplacian matrices, and are known to play a role in clustering, signal processing, and feature selection. They are also used in fast algorithms for sparse matrices. And, often for social, biological, and communication networks with very few links, their graph shapes formulate sparse matrices as for network dynamics. In a word, the number of nonzero elements in the adjacency matrix is relatively small compared to the total number of elements. Thus, a spectral property of eigenvalues by adapting a large size of a random matrix does not result in a Wigner's semicircle-like averaging [1][2][3][4][5][6]. In this paper, we treat the relevant Laplacian matrices suitable for topology of minimum links related to the features from the random matrix. Namely, the matrix is set so that the non-diagonal components of the matrix take on various random values to form a real symmetric matrix. In addition, the sum of the components in each row is fixed to be zero. It is known that, for a large dimensions of random matrix, the histogram of eigenvalues asymptotically converges a certain distribution, regardless of the types of random number generators. However, by studying the various sparse cases, it is possible to characterize the different physical factors of the networks. The networks we adopt have the same number of links (connectors) extending from the starting node to each node to spread the information; therefore, each network has the same sparsity. Based upon the analysis of disaster prevention behavior of an n-dimensional network model interpreted as a free-end spring oscillating system in our previous study, we further investigate the behavior of information transfer in detail and in an extended manner. This research can be applied to linear variations of information flows from the central facility, such as the telephone-game-like system, and network topologies constructing sparse matrices with multiple hierarchical structures. Lastly, we discuss the dynamical features of the specific sparse network with a general interpretation of the results [7][8][9][10].

II. NETWORK TOPOLOGY TO BE STUDIED

The topology of the graph structure that we have tested is shown in Figure 1. Types A, B, and C are extended from variations of the topology in which information is transmitted out of central facility. The typical system is Type Quo adopted in the previous paper. This exemplifies an application such that we can envision a system in which the first person to obtain information transfers it to the next one. It can also be interpreted to represent the transmission of information to a person belonging to a small system and disconnected from other system. Another case is a system in which information is spread from a telephone to each family member at a time, or to several partial systems from some entire system network topology.

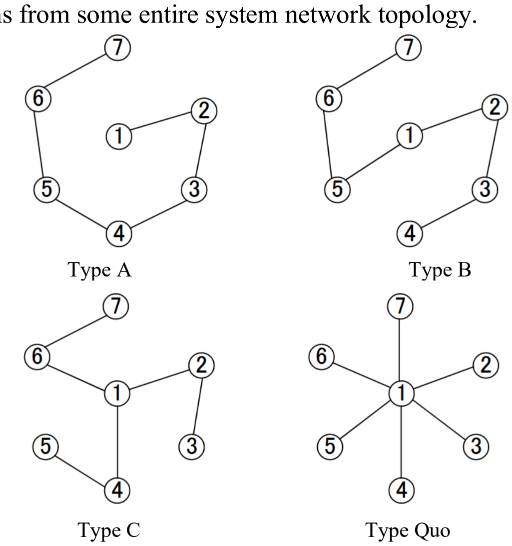


Fig. 1. Network types to be verified

The random numbers used in the experiments are Gaussian, Gamma, Poisson, Binomial, and Beta distributions. However, only those that show significant results will be presented in this paper. Although we indicate only the three extended types above, many variations of sparse networks can be found in real world. Therefore, we also check a network topology in between Type B and C to verify the trends of results.

III. Numerical experiments and theoritical perspectives

In this section, we evaluate the sparsity, which is the fraction of zero elements in the random Laplacian matrix for each type, the properties of eigenvalue distributions for types and random numbers, and the stationary property of the eigenvector corresponding to the largest eigenvalue cross-sectionally. Table 1 shows that the sparsity of matrices for a single centralized node transmitting the information is identical for all types of topologies.

TABLE I. Matrix size and sparsity

<table>
<tr><th></th><th>n=7</th><th>n=8</th><th>n=9</th><th>n=10</th><th>n=11</th></tr>
<tr><td>Type Quo</td><td rowspan="4">0.6122</td><td rowspan="4">0.6563</td><td rowspan="4">0.6914</td><td rowspan="4">0.7200</td><td rowspan="4">0.7438</td></tr>
<tr><td>Type A</td></tr>
<tr><td>Type B</td></tr>
<tr><td>Type C</td></tr>
</table>

For example, in Fig. 1, $n = 7$, so the sparsity is 0.612. where n is the number of nodes in all cases. The nodes are connected to each other with minimal links, including those in between Types B and C. The following relationship is derived for the sparsity.

$$\text{Sparseness} = \frac{n^2 - 3n + 2}{n^2} \tag{1}$$

The sparsity converges to 1 when n goes to infinity as indicated in equation (1). The asymptotic behavior of it with respect to the number of nodes is shown in Fig. 2. It can be seen a rapid convergence when n is around 100, which means that higher accuracy can be expected in the approximate solution.

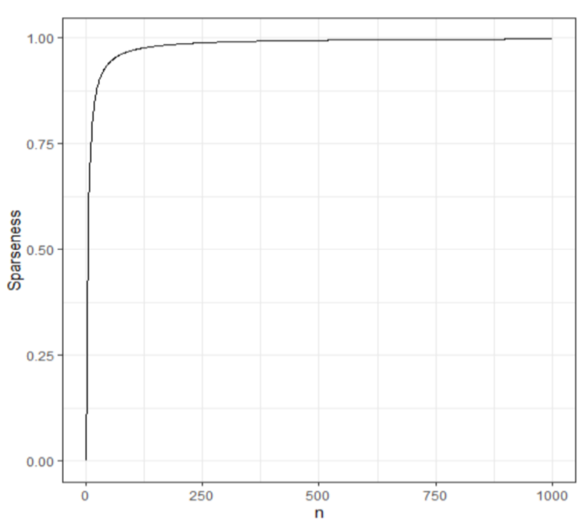


Fig. 2. Asymptotic behavior of sparsity with respect to n

Table 2 displays the eigenvalue distributions calculated from the Type Quo graph topology for a random matrix composed of Gaussian distributions with a fixed mean of 10 and three different standard deviations, as well as random numbers from the gamma, and beta distributions. It is noteworthy to mention that the shape of the eigenvalue distribution of Type Quo, a.k.a. Random Star Laplacian matrix, has the same shape as the distribution of random numbers. In fact, the eigenvalues form a similar distribution corresponding to the distribution for each random number, although our previous studies have shown that Type Quo does not obey the Wigner semicircle rule. The graphs in Fig. 1 can also be attributed to the Newton's equations of coupled vibration models of spring with free ends described in the previous study [10]. Each of the following matrices is the associated coefficient matrix. Certain components are given by random numbers, and the components x_{ij} $(j \geq i)$ of the real symmetric matrix $X = [x_{ij}]$ are constructed independently. For example, Type Quo takes the form of the following random Laplacian matrix:

$$\mathcal{L}_{ran}^{Type\ Quo} =$$

$$\begin{bmatrix} D & -|\Delta_1| & -|\Delta_2| & \cdots & \cdots & -|\Delta_{n-2}| & -|\Delta_{n-1}| \\ -|\Delta_1| & |\Delta_1| & 0 & \cdots & \cdots & 0 & 0 \\ -|\Delta_2| & 0 & |\Delta_2| & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots\ \mathbf{0} & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & 0 & \vdots \\ -|\Delta_{n-2}| & 0 & \mathbf{0}\ \ddots & \ddots & \ddots & |\Delta_{n-2}| & 0 \\ -|\Delta_{n-1}| & 0 & 0 & \cdots & \cdots & 0 & |\Delta_{n-1}| \end{bmatrix} \tag{2}$$

$$\mathrm{D} = \sum_{i=1}^{n-1} |\Delta_i|. \tag{3}$$

The matrix components $|\Delta_i|$ are mapped with the random numbers obtained from the respective probability density functions. This constructs the random Laplacian matrix to realize its graph topology. The general matrices of Type A, B, and C are shown below in (4), (6), and (8):

$$\mathcal{L}_{ran}^{Type\ A} = \begin{bmatrix} D_1 & -|\Delta_1| & & & & & \\ -|\Delta_1| & D_2 & -|\Delta_2| & & & \mathbf{0} & \\ & -|\Delta_2| & D_3 & -|\Delta_3| & & & \\ & & -|\Delta_3| & \ddots & \ddots & & \\ & \mathbf{0} & & \ddots & \ddots & -|\Delta_{n-2}| & \\ & & & & -|\Delta_{n-2}| & D_{n-1} & -|\Delta_{n-1}| \\ & & & & & -|\Delta_{n-1}| & D_n \end{bmatrix} \tag{4}$$

$$\begin{cases} D_1 = |\Delta_1| \\ D_j = \sum_{i=j-1}^{j} |\Delta_i| \quad (j = 2, 3, 4, \cdots, n-1) \\ D_n = |\Delta_{n-1}| \end{cases} \tag{5}$$

$$\mathcal{L}_{ran}^{Type\ B} =$$

$$\begin{bmatrix} D_1 & -|\Delta_1| & 0 & \cdots & -|\Delta_K| & 0 & \cdots & 0 \\ -|\Delta_1| & D_2 & -|\Delta_2| & & & & & \\ 0 & -|\Delta_2| & D_3 & -|\Delta_3| & & & \mathbf{0} & \\ \vdots & & -|\Delta_3| & \ddots & 0 & & & \\ -|\Delta_K| & & & 0 & D_K & -|\Delta_{k+1}| & & \\ 0 & \mathbf{0} & & & -|\Delta_{k+1}| & \ddots & -|\Delta_{n-2}| & \\ \vdots & & & & & -|\Delta_{n-2}| & D_{n-1} & -|\Delta_{n-1}| \\ 0 & & & & & & -|\Delta_{n-1}| & D_n \end{bmatrix} \tag{6}$$

$$\begin{cases} k = int.\left(\frac{n+1}{2}\right) \\ D_1 = |\Delta_1| + |\Delta_K| \\ D_j = \sum_{i=j-1}^{j} |\Delta_i| \ \ (2 \le j \le n-1; j \ne k) \\ D_K = |\Delta_K| + |\Delta_{k+1}| \\ D_n = |\Delta_{n-1}| \end{cases} \tag{7}$$

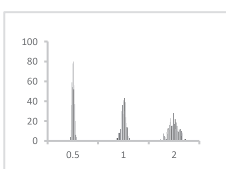
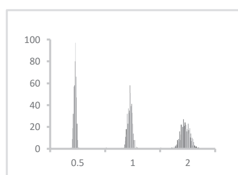
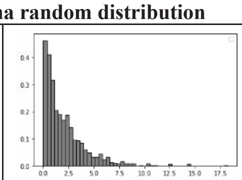
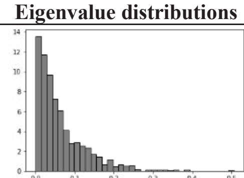
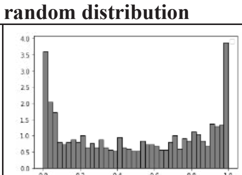
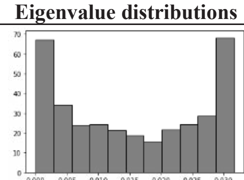
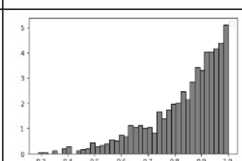
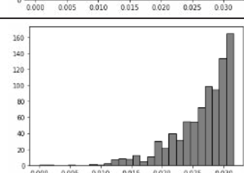
$$\mathcal{L}_{ran}^{Type\ C} =$$

$$\begin{bmatrix} D_1 & -|\Delta_1| & 0 & -|\Delta_3| & \cdots & -|\Delta_{n-3}| & \cdots & 0 \\ -|\Delta_1| & D_2 & -|\Delta_2| & & & & & \\ 0 & -|\Delta_2| & D_3 & 0 & & & \mathbf{0} & \\ -|\Delta_3| & & 0 & D_4 & -|\Delta_4| & & & \\ \vdots & & & -|\Delta_4| & \ddots & 0 & & \\ -|\Delta_{n-3}| & & \mathbf{0} & & 0 & D_{n-2} & -|\Delta_{n-2}| & \\ \vdots & & & & & -|\Delta_{n-2}| & D_{n-1} & -|\Delta_{n-1}| \\ 0 & & & & & & -|\Delta_{n-1}| & D_n \end{bmatrix} \tag{8}$$

where j is an integer greater than or equal to 1:

$$\begin{cases} D_1 = \sum_{i=1}^{int.[(n+1)/2]} |\Delta_{2i-1}| \\ D_{2j} = \sum_{i=2j-1}^{2j} |\Delta_i| \\ D_{2j+1} = |\Delta_{2j}| \\ D_n = |\Delta_{n-1}| \end{cases} \tag{9}$$

TABLE II. EIGENVALUE DISTRIBUTIONS FOR TYPE QUO WITH VARIOUS RANDOM NUMBERS ($n = 1000$)

$r^2 \approx 1.0$		
	Normal random distribution	**Eigenvalue distributions**
$\mu = 10.0$, $\sigma = \{0.5, 1.0, 2.0\}$		
	Gamma random distribution	**Eigenvalue distributions**
$\theta = 1.0$ $k = 0.5$		
	Beta random distribution	**Eigenvalue distributions**
$\alpha = 0.5$ $\beta = 0.5$		
$\alpha = 5.0$ $\beta = 1.0$		

To analyze this quantitatively, the correlation coefficients are used to see the similarity between the random distribution of linking weights and the distribution of eigenvalues. We utilize the key statistical values such as mean and standard deviation, etc. The results are summarized in Table 3.

TABLE III. CORRELATION ANALYSIS FOR THE EIGENVALUE AND RANDOM NUMBER DISTRIBUTIONS

Distributions	Average	Standard Deviation	Kurtosis	Skewness
Gaussian	0.99857	0.68576	N/A	N/A
Gamma	0.99926	0.99869	0.69245	0.90605
Beta	0.99868	0.99865	0.92358	0.98526
Chi-square	0.99987	0.99970	0.79501	0.840367

The results show that there is an extremely strong correlation between the shape of the random number and that of the eigenvalue distributions. Furthermore, Fig. 3 is a biaxial scatter plot showing the correlation between the mean values of the eigenvalues and that of the corresponding Gaussian random numbers for Type Quo. The correlation coefficient for the entire data set is that: $r^2 = 0.99857$.

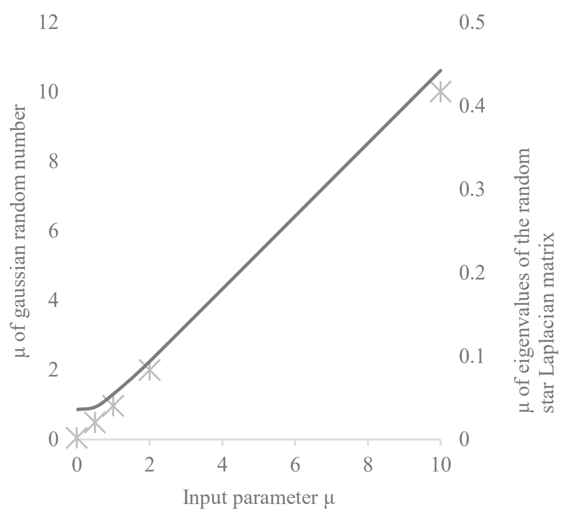


Fig. 3. Type Quo, correlation analysis of Gaussian random numbers ($\sigma = 1, \mu\{0.0, 1.5, 1.0, 2.0, 10.0\}$)

The symbol, ×, in Fig. 3 is the mean value of the Gaussian random number, and the solid curve is the mean of the eigenvalue distribution from the Laplacian matrix for Type Quo. Similar validation is performed for the gamma, Poisson, binomial, beta, and chi-square distributions, which also shows a strong correlation between shapes of the eigenvalue and random distributions. Table 4 displays the eigenvalue distributions computed from the graph topology of Type A, B, and C for random matrices composed of various random numbers with different parameters. The results depict that these eigenvalue distributions, unlike Type Quo, are not correlated with the random number distributions. It seems that the low eigenvalue spectra become more frequent for all random number distributions. However, a random distribution which generates larger values more frequently, such as the beta distribution with $\alpha = 5.0$ and $\beta = 1.0$, gives the spectra evenly distributed among medium and high eigenvalues. Moreover, in the case of Type C, the spectra are divided into two regions, one for lower and the other for higher eigenvalues.

TABLE IV. TYPE A, B, AND C EIGENVALUE DISTRIBUTIONS WITH VARIOUS RANDOM NUMBERS ($n = 1000$)

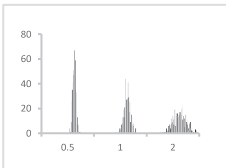
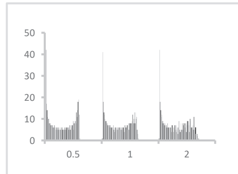
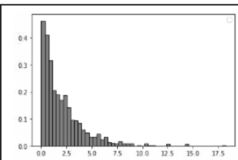
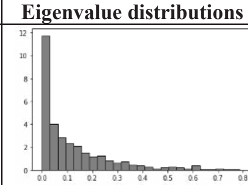
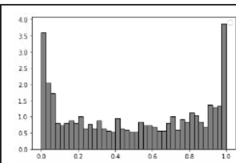
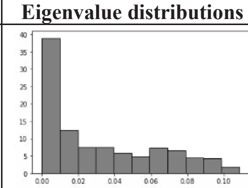
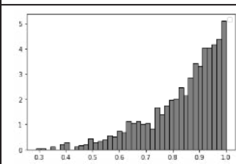
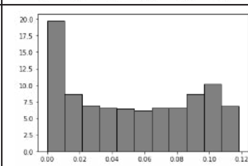
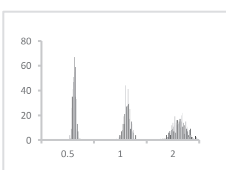
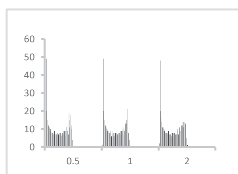
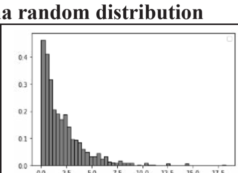
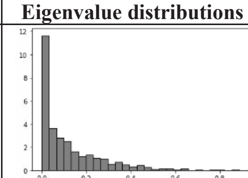
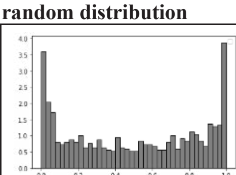
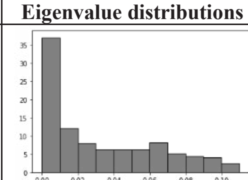
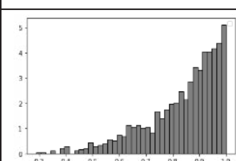
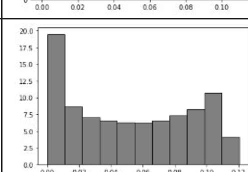
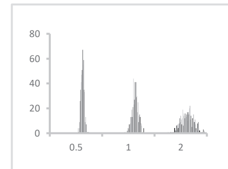
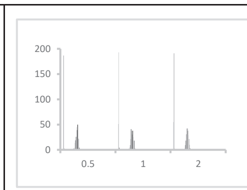
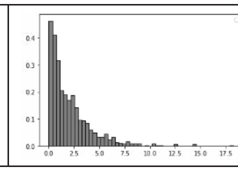
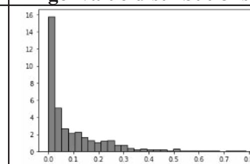
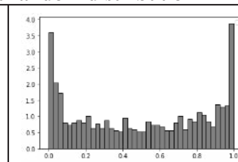
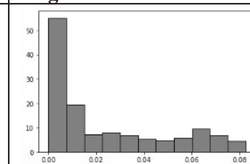
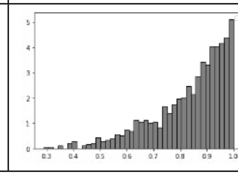
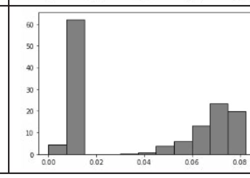
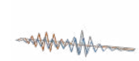
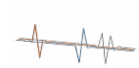
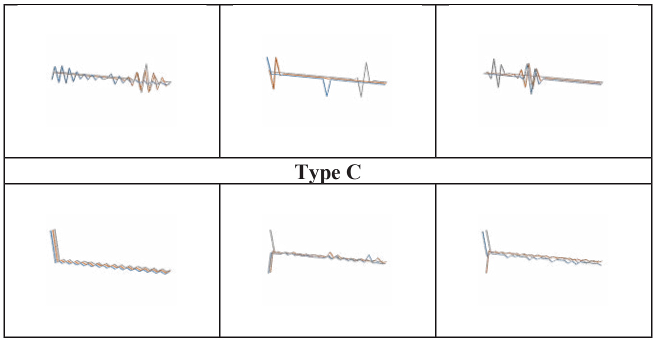
Type A, $r^2 \approx 0.0$		
Normal random distribution		**Eigenvalue distributions**
$\mu = 10.0$, $\sigma = \{0.5, 1.0, 2.0\}$		
Gamma random distribution		**Eigenvalue distributions**
$\theta = 1.0$ $k = 0.5$		
Beta random distribution		**Eigenvalue distributions**
$\alpha = 0.5$ $\beta = 0.5$		
$\alpha = 5.0$ $\beta = 1.0$		
Type B, $r^2 \approx 0.0$		
Normal random distribution		**Eigenvalue distributions**
$\mu = 10.0$, $\sigma = \{0.5, 1.0, 2.0\}$		
Gamma random distribution		**Eigenvalue distributions**
$\theta = 1.0$ $k = 0.5$		
Beta random distribution		**Eigenvalue distributions**
$\alpha = 0.5$ $\beta = 0.5$		
$\alpha = 5.0$ $\beta = 1.0$		
Type C, $r^2 \approx 0.0$		
Normal random distribution		**Eigenvalue distributions**
$\mu = 10.0$, $\sigma = \{0.5, 1.0, 2.0\}$		
Gamma random distribution		**Eigenvalue distributions**
$\theta = 1.0$ $k = 0.5$		
Beta random distribution		**Eigenvalue distributions**
$\alpha = 0.5$ $\beta = 0.5$		
$\alpha = 5.0$ $\beta = 1.0$		

Table 5 shows the eigenvectors corresponding to the maximum eigenvalues for each type computed with the random number distribution; the eigenvectors for Type Quo and Type C have a similar feature, but Type A and Type B are different from them. They exhibit a partially pulse-like shape, which indicates some specific stationary characteristics. In addition to the Gaussian random numbers, the eigenvectors of the largest eigenvalue for the beta, gamma, and Poisson distributions are compared. The eigenvectors have almost the same shape for each type. In other words, there is no difference in the shape of the eigenvectors depending on the random number distributions.

TABLE V. EIGEN VACTORS OF MAXIMUM EIGEN VALUES WITH GAUSSIAN AND OTHER RANDOM DISTRIBUTIONS ($n = 30$)

Normal random distribution: $\mu = 10.0$, $\sigma = \{0.5, 1.0, 2.0\}$	**Gamma random distribution:** $\theta = 1.0$ $k = \{0.5, 1.0, 5.0\}$	**Beta random distribution:** $\alpha = \{0.5, 1.0, 5.0\}$ $\beta = \{0.5, 1.0, 3.0\}$
Type Quo		
		
Type A		
		
Type B		



IV. Discussion and conclusion

Networks, which have the fewest number of links with a topology of centralized information transfer, take the same sparsity of their Laplacian matrices. Nevertheless, the properties of the eigenvalues and eigenvectors differ depending on the geometry of the network.

It is remarkable that the eigenvalue spectra of Type Quo (the star network) resemble those of the random number distributions. This can be interpreted as the distribution of linking weights in Type Quo is like the resonance patterns of information transmission. Then, the behavior of eigenvectors at the largest eigenvalue of Type C is similar to that of Type Quo. In other words, as the number of links from the central node increases, the eigenvectors tend to converge to the shapes that look similar to those of Type Quo. This tendency has been verified with other types of sparse networks.

On the other hand, for Type A and Type B, the behavior of the eigenvectors at the maximum eigenvalue is oscillatory, which can be interpreted as an excitation of relative information to make bias in some links of the information transmission. The eigenvalue distribution of Type Quo is highly correlated with that of random numbers as shown in Fig. 3. However, the eigenvalue distributions of Type A, B, and C are not similar to those of the random numbers, but the spectra of the middle and high eigenvalues vary with the distributions of the random numbers slightly. The common feature of Type A, B, and C is that the eigenvalues spectra tend to be concentrated near zero. This property is considered to reflect the nature of eigenvalue distributions obtained from general random sparse matrices [11]. It is interesting to note that the eigenvectors corresponding to the largest eigenvalues such that Type C exhibit stationary properties similar to those of Type Quo, while the eigenvalue distribution properties are different from those of Type Quo. Namely, the eigenvector shapes tend to converge, but the eigenvalues do not.

In this study, we have found that the sparsity of matrices has nothing to do with properties of eigenvalue distributions and eigenvectors in centralized network topologies. Then, the linkages with fewer than three branches from the center and those with many branches besides Type Quo have similar shapes of eigenvalue distributions, but eigenvectors. The results show that a notable difference appears in the eigenvectors, i.e., the stationary state of the information. This research may provide clues to analyze how the shape of the infrastructure of information transfer affects its efficiency and effectiveness. The further developments will be expected in the future.

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Appendix

Algorithm to produce the results of random Laplacian matrices:

1. Determine the size of the matrix by selecting n (e.g., $n = 100$)
2. Set the parameters for the random numbers out of probability density functions (PDF). (e.g., Normal random distribution: $\mu = 0.0, \sigma = 1.0$)
3. Each element $A[i, j]$ is independently sampled from the PDF in procedure 1
4. Construct the Laplacian matrix, $\mathcal{L}_{ran}$: The elements of $\mathcal{L}_{ran}[i, j]$ must be symmetric, and then the sum of each row has to be zero.
5. Compute the eigenvalues and eigenvectors.
6. Normalize the eigenvalue spectra with dividing by $\sqrt{n}$. Arrange for the visualization of data: Obtain the array of eigenvalues (excluding the 0 eigenvalue.) Create histograms of eigenvalues, and plots of eigenvectors.

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